

Applications of Toponogov's comparison theorems for open triangles^{*†}

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Abstract

Recently we generalized Toponogov's comparison theorem to a complete Riemannian manifold with smooth convex boundary, where a geodesic triangle was replaced by an open (geodesic) triangle standing on the boundary of the manifold, and a model surface was replaced by the universal covering surface of a cylinder of revolution with totally geodesic boundary. The aim of this article is to prove splitting theorems of two types as an application. Moreover, we establish a weaker version of our Toponogov comparison theorem for open triangles, because the weaker version is quite enough to prove one of the splitting theorems.

1 Introduction

Words have fully expressed a matter of great importance for Toponogov's comparison theorem. However that may be, we can not stop telling the importance in Riemannian geometry. The comparison theorem has played a vital role in the comparison geometry, that is, the theorem gives us some techniques originating from Euclidean geometry. Such techniques, drawing a circle or a geodesic polygon, and joining two points by a minimal geodesic segment, are very powerful in the geometry. One may find concrete examples of such techniques in proofs of the maximal diameter theorem and the splitting theorem by Toponogov ([T1], [T2]), the structure theorem with positive sectional curvature by Gromoll and Meyer ([GM]), the soul theorem with non-negative sectional curvature by Cheeger and Gromoll ([CG]), the diameter sphere theorem by Grove and Shiohama ([GS]), etc.

From the standpoint of the radial curvature geometry, we very recently generalized the Toponogov comparison theorem to a complete Riemannian manifold with smooth convex boundary, where a geodesic triangle was replaced by an open (geodesic) triangle standing on the boundary of the manifold, and a model surface was replaced by the universal covering surface of a cylinder of revolution with totally geodesic boundary ([KT2, Theorem 8.4], which will be stated as Theorem 2.5 in this article).

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The aim of our article is to prove splitting theorems of two types as an application of Toponogov's comparison theorem for open triangles and a weaker version of the comparison theorem (Theorem 2.12), respectively. The weaker version will be proved in this article.

Now we will introduce the radial curvature geometry for manifolds with boundary: We first introduce our model, which will be later employed as a reference surface of comparison theorems in complete Riemannian manifolds with boundary. Let $\widetilde{M} := (\mathbb{R}, d\tilde{x}^2) \times_m (\mathbb{R}, d\tilde{y}^2)$ be a warped product of two 1-dimensional Euclidean lines $(\mathbb{R}, d\tilde{x}^2)$ and $(\mathbb{R}, d\tilde{y}^2)$, where the warping function $m : \mathbb{R} \rightarrow (0, \infty)$ is a positive smooth function satisfying $m(0) = 1$ and $m'(0) = 0$. Then we call

$$\widetilde{X} := \left\{ \tilde{p} \in \widetilde{M} \mid \tilde{x}(\tilde{p}) \geq 0 \right\}$$

a *model surface*. Since $m'(0) = 0$, the boundary $\partial\widetilde{X} := \{\tilde{p} \in \widetilde{X} \mid \tilde{x}(\tilde{p}) = 0\}$ of \widetilde{X} is *totally geodesic*. The metric \tilde{g} of \widetilde{X} is expressed as

$$\tilde{g} = d\tilde{x}^2 + m(\tilde{x})^2 d\tilde{y}^2 \quad (1.1)$$

on $[0, \infty) \times \mathbb{R}$. The function $G \circ \tilde{\mu} : [0, \infty) \rightarrow \mathbb{R}$ is called the *radial curvature function* of \widetilde{X} , where we denote by G the Gaussian curvature of \widetilde{X} , and by $\tilde{\mu}$ any ray emanating perpendicularly from $\partial\widetilde{X}$ (Notice that such a $\tilde{\mu}$ will be called a $\partial\widetilde{X}$ -ray). Remark that $m : [0, \infty) \rightarrow \mathbb{R}$ satisfies the differential equation $m''(t) + G(\tilde{\mu}(t))m(t) = 0$ with initial conditions $m(0) = 1$ and $m'(0) = 0$. Note that the n -dimensional model surfaces are defined similarly, and, as seen in [KK], we may completely classify them by taking half spaces of spaces in [MS, Theorem 1.1].

Hereafter, let $(X, \partial X)$ denote a complete Riemannian n -dimensional manifold X with smooth boundary ∂X . We say that ∂X is *convex*, if all eigenvalues of the shape operator A_ξ of ∂X are non-negative in the inward vector ξ normal to ∂X . Notice that our sign of A_ξ differs from [S]. That is, for each $p \in \partial X$ and $v \in T_p \partial X$, $A_\xi(v) = -(\nabla_v N)^\top$ holds. Here, we denote by N a local extension of ξ , and by ∇ the Riemannian connection on X .

For a positive constant ℓ , a unit speed geodesic segment $\mu : [0, \ell] \rightarrow X$ emanating from ∂X is called a ∂X -segment, if $d(\partial X, \mu(t)) = t$ on $[0, \ell]$. If $\mu : [0, \ell] \rightarrow X$ is a ∂X -segment for all $\ell > 0$, we call μ a ∂X -ray. Here, we denote by $d(\partial X, \cdot)$ the distance function to ∂X induced from the Riemannian structure of X . Notice that a ∂X -segment is orthogonal to ∂X by the first variation formula, and so a ∂X -ray is too.

$(X, \partial X)$ is said to have the *radial curvature (with respect to ∂X) bounded from below by that of $(\widetilde{X}, \partial\widetilde{X})$* if, for every ∂X -segment $\mu : [0, \ell] \rightarrow X$, the sectional curvature K_X of X satisfies

$$K_X(\sigma_t) \geq G(\tilde{\mu}(t))$$

for all $t \in [0, \ell)$ and all 2-dimensional linear spaces σ_t spanned by $\mu'(t)$ and a tangent vector to X at $\mu(t)$. For example, if the Riemannian metric of \widetilde{X} is $d\tilde{x}^2 + d\tilde{y}^2$, or $d\tilde{x}^2 + \cosh^2(\tilde{x}) d\tilde{y}^2$, then $G(\tilde{\mu}(t)) = 0$, or $G(\tilde{\mu}(t)) = -1$, respectively. Furthermore, **the radial curvature may change signs wildly**. Examples of a model surfaces admitting such a crazy behavior of radial curvature are found in [TK, Theorems 1.3 and 4.1].

Our main theorems in this article are now stated as follows:

Theorem 1.1 *Let $(X, \partial X)$ be a complete non-compact connected Riemannian manifold X with smooth convex boundary ∂X whose radial curvature is bounded from below by that of a model surface $(\tilde{X}, \partial\tilde{X})$ with its metric (1.1). Assume that X admits at least one ∂X -ray.*

(ST-1) *If $(\tilde{X}, \partial\tilde{X})$ satisfies*

$$\int_0^\infty \frac{1}{m(t)^2} dt = \infty,$$

then X is isometric to $[0, \infty) \times_m \partial X$. In particular, ∂X is the soul of X , and the number of connected components of ∂X is one.

(ST-2) *If $(\tilde{X}, \partial\tilde{X})$ satisfies $\liminf_{t \rightarrow \infty} m(t) = 0$, then X is diffeomorphic to $[0, \infty) \times \partial X$. In particular, the number of connected components of ∂X is one.*

Toponogov's comparison theorem for open triangles in a weak form (Theorem 2.12) will be applied in the proof of Theorem 1.1 (see Section 4). The assumption on the existence of a ∂X -ray is very natural, because we may find at least one ∂X -ray if ∂X is compact. If the model \tilde{X} is Euclidean (i.e., $m \equiv 1$), then the (ST-1) holds. Hence, Theorem 1.1 extends one of Burago and Zalgaller' splitting theorems to a wider class of metrics than those described in [BZ, Theorem 5.2.1], i.e., we mean that they assumed that sectional curvature is **non-negative everywhere**.

Theorem 1.2 *Let $(X, \partial X)$ be a complete connected Riemannian manifold X with disconnected smooth compact convex boundary ∂X whose radial curvature is bounded from below by 0. Then, X is isometric to $[0, \ell] \times \partial X_1$ with Euclidean product metric of $[0, \ell]$ and ∂X_1 , where ∂X_1 denotes a connected component of ∂X . In particular, ∂X_1 is the soul of X .*

Toponogov's comparison theorem for open triangles (Theorem 2.5) will be applied in the proof of Theorem 1.2 (see Section 5). Notice that non-negative radial curvature **does not always mean** non-negative sectional curvature (cf. [KT1, Example 5.6]). Although Theorem 1.2 extends one of Burago and Zalgaller' splitting theorems to a wider class of metrics than those described in [BZ, Theorem 5.2.1], Ichida [I] and Kasue [K] obtain the same conclusion of the theorem under weaker assumptions, i.e., the mean curvature (with respect to the inner normal direction) of boundary are non-negative, and that Ricci curvature is non-negative everywhere.

In the following sections, all geodesics will be normalized, unless otherwise stated.

2 Toponogov's Theorems for Open Triangles

Throughout this section, let $(X, \partial X)$ denote a complete connected Riemannian manifold X with smooth **convex** boundary ∂X whose radial curvature is bounded from below by that of a model surface $(\tilde{X}, \partial\tilde{X})$ with its metric (1.1).

Definition 2.1 (Open Triangles) For any fixed two points $p, q \in X \setminus \partial X$, an *open triangle*

$$\text{OT}(\partial X, p, q) = (\partial X, p, q; \gamma, \mu_1, \mu_2)$$

in X is defined by two ∂X -segments $\mu_i : [0, \ell_i] \rightarrow X$, $i = 1, 2$, a minimal geodesic segment $\gamma : [0, d(p, q)] \rightarrow X$, and ∂X such that $\mu_1(\ell_1) = \gamma(0) = p$, $\mu_2(\ell_2) = \gamma(d(p, q)) = q$.

Remark 2.2 In this article, whenever an open triangle $\text{OT}(\partial X, p, q) = (\partial X, p, q; \gamma, \mu_1, \mu_2)$ in X is given, $(\partial X, p, q; \gamma, \mu_1, \mu_2)$, as a symbol, always means that the minimal geodesic segment γ is the opposite side to ∂X emanating from p to q , and that the ∂X -segments μ_1, μ_2 are sides emanating from ∂X to p, q , respectively.

Definition 2.3 We call the set $\tilde{X}(\theta) := \tilde{y}^{-1}((0, \theta))$ a sector in \tilde{X} for each constant number $\theta > 0$.

Remark 2.4 Since a map $(\tilde{p}, \tilde{q}) \rightarrow (\tilde{p}, \tilde{q} + c)$, $c \in \mathbb{R}$, over \tilde{X} is an isometry, a sector $\tilde{X}(\theta)$ is isometric to $\tilde{y}^{-1}(c, c + \theta)$ for all $c \in \mathbb{R}$.

Toponogov's comparison theorem for open triangles is stated as follows:

Theorem 2.5 ([KT2, Theorem 8.4]) *Let $(X, \partial X)$ be a complete connected Riemannian manifold X with smooth convex boundary ∂X whose radial curvature is bounded from below by that of a model surface $(\tilde{X}, \partial \tilde{X})$ with its metric (1.1). Assume that \tilde{X} admits a sector $\tilde{X}(\theta_0)$ which has no pair of cut points. Then, for every open triangle $\text{OT}(\partial X, p, q) = (\partial X, p, q; \gamma, \mu_1, \mu_2)$ in X with $d(\mu_1(0), \mu_2(0)) < \theta_0$, there exists an open triangle $\text{OT}(\partial \tilde{X}, \tilde{p}, \tilde{q}) = (\partial \tilde{X}, \tilde{p}, \tilde{q}; \tilde{\gamma}, \tilde{\mu}_1, \tilde{\mu}_2)$ in $\tilde{X}(\theta_0)$ such that*

$$d(\partial \tilde{X}, \tilde{p}) = d(\partial X, p), \quad d(\tilde{p}, \tilde{q}) = d(p, q), \quad d(\partial \tilde{X}, \tilde{q}) = d(\partial X, q) \quad (2.1)$$

and that

$$\angle p \geq \angle \tilde{p}, \quad \angle q \geq \angle \tilde{q}, \quad d(\mu_1(0), \mu_2(0)) \geq d(\tilde{\mu}_1(0), \tilde{\mu}_2(0)). \quad (2.2)$$

Furthermore, if $d(\mu_1(0), \mu_2(0)) = d(\tilde{\mu}_1(0), \tilde{\mu}_2(0))$ holds, then

$$\angle p = \angle \tilde{p}, \quad \angle q = \angle \tilde{q}$$

hold. Here $\angle p$ denotes the angle between two vectors $\gamma'(0)$ and $-\mu_1'(d(\partial X, p))$ in $T_p X$.

Remark 2.6 In Theorem 2.5, we do not assume that ∂X is connected. Moreover, the opposite side γ of $\text{OT}(\partial X, p, q)$ does not meet ∂X (see [KT2, Lemma 6.1]). In [MS], they treat a pair (M, N) of a complete connected Riemannian manifold M and a compact connected totally geodesic hypersurface N of M such that the radial curvature with respect to N is bounded from below by that of the model $((a, b) \times_m N, N)$, where (a, b) denotes an interval, in their sense. Note that the radial curvature with respect to N is bounded from below by that of our model $([0, \infty), d\tilde{x}^2) \times_m (\mathbb{R}, d\tilde{y}^2)$, if it is bounded from below by that of their model $((a, b) \times_m N, N)$. Thus, Theorem 2.5 is **applicable to** the pair (M, N) .

In the following, we will prove the Toponogov comparison theorem for open triangles in a weak form (Theorem 2.12), where we do not demand any assumption on a sector. To do so, we need to introduce definitions and a key lemma:

Definition 2.7 (Generalized open triangles) A generalized open triangle

$$\text{GOT}(\partial\tilde{X}, \hat{p}, \hat{q}) = (\partial\tilde{X}, \hat{p}, \hat{q}; \hat{\gamma}, \hat{\mu}_1, \hat{\mu}_2)$$

in \tilde{X} is defined by two $\partial\tilde{X}$ -segments $\hat{\mu}_i : [0, \ell_i] \rightarrow \tilde{X}$, $i = 1, 2$, and a geodesic segment $\hat{\gamma}$ emanating from \hat{p} to \hat{q} such that $\hat{\mu}_1(\ell_1) = \hat{\gamma}(0) = \hat{p}$, $\hat{\mu}_2(\ell_2) = \hat{\gamma}(d(\hat{p}, \hat{q})) = \hat{q}$, and that $\hat{\gamma}$ is a shortest arc joining \hat{p} to \hat{q} in the compact domain bounded by $\hat{\mu}_1$, $\hat{\mu}_2$, and $\hat{\gamma}$.

Definition 2.8 (The injectivity radius) The *injectivity radius* $\text{inj}(\tilde{p})$ of a point $\tilde{p} \in \tilde{X}$ is the supremum of $r > 0$ such that, for any point $\tilde{q} \in \tilde{X}$ with $d(\tilde{p}, \tilde{q}) < r$, there exists a unique minimal geodesic segment joining \tilde{p} to \tilde{q} .

Remark 2.9 For each point $\tilde{p} \in \tilde{X} \setminus \partial\tilde{X}$, $\text{inj}(\tilde{p}) > d(\partial\tilde{X}, \tilde{p})$ holds, if \tilde{p} is sufficiently close to $\partial\tilde{X}$.

Definition 2.10 (Thin Open Triangle) An open triangle $\text{OT}(\partial X, p, q)$ in X is called a *thin open triangle*, if

(TOT-1) the opposite side γ of $\text{OT}(\partial X, p, q)$ to ∂X emanating from p to q is contained in a normal convex neighborhood in $X \setminus \partial X$, and

(TOT-2) $L(\gamma) < \text{inj}(\tilde{q}_s)$ for all $s \in [0, d(p, q)]$,

where $L(\gamma)$ denotes the length of γ , and \tilde{q}_s denotes a point in \tilde{X} with $d(\partial\tilde{X}, \tilde{q}_s) = d(\partial X, \gamma(s))$ for each $s \in [0, d(p, q)]$.

Then, we have the key lemma to prove the weaker version of Toponogov's comparison theorem for open triangles.

Lemma 2.11 ([KT2, Lemma 5.8]) *For every thin open triangle $\text{OT}(\partial X, p, q)$ in X , there exists an open triangle $\text{OT}(\partial\tilde{X}, \tilde{p}, \tilde{q})$ in \tilde{X} such that*

$$d(\partial\tilde{X}, \tilde{p}) = d(\partial X, p), \quad d(\tilde{p}, \tilde{q}) = d(p, q), \quad d(\partial\tilde{X}, \tilde{q}) = d(\partial X, q) \quad (2.3)$$

and that

$$\angle p \geq \angle \tilde{p}, \quad \angle q \geq \angle \tilde{q}. \quad (2.4)$$

Now, the weaker version of Toponogov's comparison theorem for open triangles is stated as follows:

Theorem 2.12 *Let $(X, \partial X)$ be a complete connected Riemannian manifold X with smooth convex boundary ∂X whose radial curvature is bounded from below by that of a model surface $(\tilde{X}, \partial\tilde{X})$. Then, for every open triangle $\text{OT}(\partial X, p, q) = (\partial X, p, q; \gamma, \mu_1, \mu_2)$ in X ,*

there exists a generalized open triangle $\text{GOT}(\partial\tilde{X}, \hat{p}, \hat{q}) = (\partial\tilde{X}, \hat{p}, \hat{q}; \hat{\gamma}, \hat{\mu}_1, \hat{\mu}_2)$ in \tilde{X} such that

$$d(\partial\tilde{X}, \hat{p}) = d(\partial X, p), \quad d(\partial\tilde{X}, \hat{q}) = d(\partial X, q), \quad (2.5)$$

and

$$d(\partial X, q) - d(\partial X, p) \leq d(\hat{p}, \hat{q}) \leq L(\hat{\gamma}) \leq d(p, q), \quad (2.6)$$

and that

$$\angle p \geq \angle \hat{p}, \quad \angle q \geq \angle \hat{q}. \quad (2.7)$$

Here $L(\hat{\gamma})$ denotes the length of $\hat{\gamma}$.

Proof. Let $s_0 := 0 < s_1 < \dots < s_{k-1} < s_k := d(p, q)$ be a subdivision of $[0, d(p, q)]$ such that, for each $i \in \{1, \dots, k\}$, the open triangle $\text{OT}(\partial X, \gamma(s_{i-1}), \gamma(s_i))$ is thin. It follows from Lemma 2.11 that, for each triangle $\text{OT}(\partial X, \gamma(s_{i-1}), \gamma(s_i))$, there exists an open triangle $\tilde{\Delta}_i := \text{OT}(\partial\tilde{X}, \tilde{\gamma}(s_{i-1}), \tilde{\gamma}(s_i))$ in \tilde{X} such that

$$d(\partial\tilde{X}, \tilde{\gamma}(s_{i-1})) = d(\partial X, \gamma(s_{i-1})), \quad (2.8)$$

$$d(\tilde{\gamma}(s_{i-1}), \tilde{\gamma}(s_i)) = d(\gamma(s_{i-1}), \gamma(s_i)), \quad (2.9)$$

$$d(\partial\tilde{X}, \tilde{\gamma}(s_i)) = d(\partial X, \gamma(s_i)), \quad (2.10)$$

and that

$$\angle(\partial X, \gamma(s_{i-1}), \gamma(s_i)) \geq \angle(\partial\tilde{X}, \tilde{\gamma}(s_{i-1}), \tilde{\gamma}(s_i)), \quad (2.11)$$

$$\angle(\partial X, \gamma(s_i), \gamma(s_{i-1})) \geq \angle(\partial\tilde{X}, \tilde{\gamma}(s_i), \tilde{\gamma}(s_{i-1})). \quad (2.12)$$

Here $\angle(\partial X, \gamma(s_{i-1}), \gamma(s_i))$ denotes the angle between two sides joining $\gamma(s_{i-1})$ to ∂X and $\gamma(s_i)$ forming the triangle $\text{OT}(\partial X, \gamma(s_{i-1}), \gamma(s_i))$. Under this situation, draw $\tilde{\Delta}_1 = \text{OT}(\partial\tilde{X}, \tilde{p}, \tilde{\gamma}(s_1))$ in \tilde{X} satisfying (2.8), (2.9), (2.10), (2.11), (2.12) for $i = 1$. Inductively, we draw an open triangle $\tilde{\Delta}_{i+1} = \text{OT}(\partial\tilde{X}, \tilde{\gamma}(s_i), \tilde{\gamma}(s_{i+1}))$ in \tilde{X} , which is adjacent to $\tilde{\Delta}_i$ so as to have the $\partial\tilde{X}$ -segment to $\tilde{\gamma}(s_i)$ as a common side. Since

$$\angle(\partial X, \gamma(s_i), \gamma(s_{i-1})) + \angle(\partial X, \gamma(s_i), \gamma(s_{i+1})) = \pi,$$

for each $i = 1, 2, \dots, k-1$, we get, by (2.11) and (2.12),

$$\angle(\partial\tilde{X}, \tilde{\gamma}(s_i), \tilde{\gamma}(s_{i-1})) + \angle(\partial\tilde{X}, \tilde{\gamma}(s_i), \tilde{\gamma}(s_{i+1})) \leq \pi \quad (2.13)$$

and

$$\angle p \geq \angle(\partial\tilde{X}, \tilde{\gamma}(s_0), \tilde{\gamma}(s_1)), \quad \angle q \geq \angle(\partial\tilde{X}, \tilde{\gamma}(s_k), \tilde{\gamma}(s_{k-1})). \quad (2.14)$$

Then, we get a domain \mathcal{D} bounded by two $\partial\tilde{X}$ -segments $\tilde{\mu}_0, \tilde{\mu}_k$ to $\tilde{\gamma}(s_0), \tilde{\gamma}(s_k)$, respectively, and $\tilde{\eta}$, where $\tilde{\eta}$ denotes the broken geodesic consisting of the opposite sides of $\tilde{\Delta}_i$ ($i = 1, 2, \dots, k$) to $\partial\tilde{X}$. Since the domain \mathcal{D} is locally convex by (2.13), there exists a minimal geodesic segment $\hat{\gamma}$ in the closure of \mathcal{D} joining $\tilde{\gamma}(s_0)$ to $\tilde{\gamma}(s_k)$. From (2.14), it is clear that the generalized open triangle $(\partial\tilde{X}, \tilde{\gamma}(s_0), \tilde{\gamma}(s_k); \hat{\gamma}, \tilde{\mu}_0, \tilde{\mu}_k)$ has the required properties in our theorem. \square

3 Definitions and notations for Sections 4 and 5

Throughout this section, let $(X, \partial X)$ denote a complete connected Riemannian manifold X with smooth boundary ∂X . Our purpose of this section is to recall the definitions of ∂X -Jacobi fields, focal loci of ∂X , and cut loci of ∂X , which will appear in Sections 4 and 5.

Definition 3.1 (∂X -Jacobi field) Let $\mu : [0, \infty) \rightarrow X$ be a unit speed geodesic emanating perpendicularly from ∂X . A Jacobi field $J_{\partial X}$ along μ is called a ∂X -Jacobi field, if $J_{\partial X}$ satisfies $J_{\partial X}(0) \in T_{\mu(0)}\partial X$ and $J'_{\partial X}(0) + A_{\mu'(0)}(J_{\partial X}(0)) \in (T_{\mu(0)}\partial X)^\perp$. Here J' denotes the covariant derivative of J along μ , and $A_{\mu'(0)}$ denotes the shape operator of ∂X .

Definition 3.2 (Focal locus of ∂X) A point $\mu(t_0)$, $t_0 \neq 0$, is called a *focal point* of ∂X along a unit speed geodesic $\mu : [0, \infty) \rightarrow X$ emanating perpendicularly from ∂X , if there exists a non-zero ∂X -Jacobi field $J_{\partial X}$ along μ such that $J_{\partial X}(t_0) = 0$. The *focal locus* $\text{Foc}(\partial X)$ of ∂X is the union of the focal points of ∂X along all of the unit speed geodesics emanating perpendicularly from ∂X .

Definition 3.3 (Cut locus of ∂X) Let $\mu : [0, \ell_0] \rightarrow X$ be a ∂X -segment. The end point $\mu(\ell_0)$ of $\mu([0, \ell_0])$ is called a *cut point* of ∂X along μ , if any extended geodesic $\bar{\mu} : [0, \ell_1] \rightarrow X$ of μ , $\ell_1 > \ell_0$, is not a ∂X -segment anymore. The *cut locus* $\text{Cut}(\partial X)$ of ∂X is the union of the cut points of ∂X along all of the ∂X -segments.

4 Proof of Theorem 1.1

From the similar argument in the proof of [ST, Lemma 3.1], one may prove

Lemma 4.1 *Let*

$$\begin{aligned} f''(t) + K(t)f(t) &= 0, & f(0) &= 1, & t &\in [0, \infty), \\ m''(t) + G(t)m(t) &= 0, & m(0) &= 1, & m'(0) &= 0, & t &\in [0, \infty), \end{aligned}$$

be two ordinary differential equations with $K(t) \geq G(t)$ on $[0, \infty)$.

(L-1) *If $f > 0$ on $(0, \infty)$, $f'(0) = 0$, and*

$$\int_0^\infty \frac{1}{m(t)^2} dt = \infty,$$

then $K(t) = G(t)$ on $[0, \infty)$.

(L-2) *If $m > 0$ on $(0, \infty)$, $f'(0) < 0$, and*

$$\int_0^\infty \frac{1}{m(t)^2} dt = \infty,$$

then there exists $t_0 \in (0, \infty)$ such that $f > 0$ on $[0, t_0)$ and $f(t_0) = 0$.

Hereafter, let $(X, \partial X)$ be a complete non-compact connected Riemannian n -manifold X with smooth **convex** boundary ∂X whose radial curvature is bounded from below by that of a model surface $(\tilde{X}, \partial \tilde{X})$ with its metric (1.1). Moreover, we denote by

$$\mathcal{I}_{\partial X}^\ell(V, W) := I_\ell(V, W) - \langle A_{\mu'(0)}(V(0)), W(0) \rangle$$

the index form with respect to a ∂X -segment $\mu : [0, \ell] \rightarrow X$ for piecewise C^∞ vector fields V, W along μ , where we set

$$I_\ell(V, W) := \int_0^\ell \{ \langle V', W' \rangle - \langle R(\mu', V)\mu', W \rangle \} dt,$$

which is a symmetric bilinear form. Furthermore,

we assume that X admits at least one ∂X -ray.

By Lemma 4.1, we have

Lemma 4.2 *Let $\mu : [0, \infty) \rightarrow X$ be a ∂X -ray. If $(\tilde{X}, \partial \tilde{X})$ satisfies*

$$\int_0^\infty \frac{1}{m(t)^2} dt = \infty,$$

then, $\mu(0)$ is the geodesic point in ∂X , i.e., the second fundamental form vanishes at the point.

Proof. Let E be a unit parallel vector field along μ such that

$$A_{\mu'(0)}(E(0)) = \lambda E(0), \tag{4.1}$$

$$E(t) \perp \mu'(t). \tag{4.2}$$

Here λ denotes an eigenvalue of the shape operator $A_{\mu'(0)}$ of ∂X . Since ∂X is convex, $\lambda \geq 0$ holds. Consider a smooth vector field $Y(t) := f(t)E(t)$ along μ satisfying

$$f''(t) + K_X(\mu'(t), E(t))f(t) = 0,$$

with initial conditions

$$f(0) = 1, \quad f'(0) = -\lambda. \tag{4.3}$$

Here $K_X(\mu'(t), E(t))$ denotes the sectional curvature with respect to the 2-dimensional linear space spanned by $\mu'(t)$ and $E(t)$ at $\mu(t)$. Notice that Y satisfies $Y(0) \in T_{\mu(0)}\partial X$ and $Y'(0) + A_{\mu'(0)}(Y(0)) = 0 \in (T_{\mu(0)}\partial X)^\perp$, by (4.1), (4.2), and (4.3). Suppose that $\lambda > 0$. Since $f'(0) < 0$ and

$$\int_0^\infty \frac{1}{m(t)^2} dt = \infty,$$

it follows from (L-2) in Lemma 4.1 that there exists $t_0 \in (0, \infty)$ such that $f > 0$ on $[0, t_0)$ and

$$f(t_0) = 0, \tag{4.4}$$

i.e.,

$$Y(t) \neq 0, \quad t \in [0, t_0) \quad (4.5)$$

and $Y(t_0) = 0$. Since $\langle R(\mu'(t), Y(t))\mu'(t), Y(t) \rangle = f(t)^2 \langle R(\mu'(t), E(t))\mu'(t), E(t) \rangle = -f''(t)f(t)$, we have, by (4.3) and (4.4),

$$I_{t_0}(Y, Y) = \int_0^{t_0} \frac{d}{dt}(ff')dt = f(t_0)f'(t_0) - f(0)f'(0) = \lambda. \quad (4.6)$$

Thus, by (4.1), (4.3), and (4.6),

$$\mathcal{I}_{\partial X}^{t_0}(Y, Y) = I_{t_0}(Y, Y) - \langle A_{\mu'(0)}(Y(0)), Y(0) \rangle = \lambda - \lambda = 0. \quad (4.7)$$

On the other hand, since ∂X has no focal point along μ , for any non-zero vector field Z along μ satisfying $Z(0) \in T_{\mu(0)}\partial X$ and $Z(t_0) = 0$,

$$\mathcal{I}_{\partial X}^{t_0}(Z, Z) > 0 \quad (4.8)$$

holds (cf. Lemma 2.9 in [S, Chapter III]). Thus, by (4.7) and (4.8), $Y \equiv 0$ on $[0, t_0]$. This is a contradiction to (4.5). Therefore, $\lambda = 0$, i.e., $\mu(0)$ is the geodesic point in ∂X . \square

Here we want to go over some fundamental tools on $(\tilde{X}, \partial\tilde{X})$: A unit speed geodesic $\tilde{\gamma} : [0, a) \rightarrow \tilde{X}$ ($0 < a \leq \infty$) is expressed by $\tilde{\gamma}(s) = (\tilde{x}(\tilde{\gamma}(s)), \tilde{y}(\tilde{\gamma}(s))) =: (\tilde{x}(s), \tilde{y}(s))$. Then, there exists a non-negative constant ν depending only on $\tilde{\gamma}$ such that

$$\nu = m(\tilde{x}(s))^2 |\tilde{y}'(s)| = m(\tilde{x}(s)) \sin \angle(\tilde{\gamma}'(s), (\partial/\partial\tilde{x})_{\tilde{\gamma}(s)}). \quad (4.9)$$

This (4.9) is a famous formula – the *Clairaut relation*. The constant ν is called the *Clairaut constant* of $\tilde{\gamma}$. Remark that, by (4.9), $\nu > 0$ if and only if $\tilde{\gamma}$ is not a $\partial\tilde{X}$ -ray, or its subarc. Since $\tilde{\gamma}$ is unit speed, we have, by (4.9),

$$\tilde{x}'(s) = \pm \frac{\sqrt{m(\tilde{x}(s))^2 - \nu^2}}{m(\tilde{x}(s))}. \quad (4.10)$$

By (4.10), we see that $\tilde{x}'(s) = 0$ if and only if $m(\tilde{x}(s)) = \nu$. Moreover, by (4.10), we have that, for a unit speed geodesic $\tilde{\gamma}(s) = (\tilde{x}(s), \tilde{y}(s))$, $s_1 \leq s \leq s_2$, with the Clairaut constant ν ,

$$s_2 - s_1 = \phi(\tilde{x}'(s)) \int_{\tilde{x}(s_1)}^{\tilde{x}(s_2)} \frac{m(t)}{\sqrt{m(t)^2 - \nu^2}} dt, \quad (4.11)$$

if $\tilde{x}'(s) \neq 0$ on (s_1, s_2) . Here, $\phi(\tilde{x}'(s))$ denotes the sign of $\tilde{x}'(s)$. Furthermore, we have a lemma with respect to the length $L(\tilde{\gamma})$ of $\tilde{\gamma}$:

Lemma 4.3 *Let $\tilde{\gamma} : [0, s_0] \rightarrow \tilde{X} \setminus \partial\tilde{X}$ denote a unit speed geodesic segment with Clairaut constant ν . Then, $L(\tilde{\gamma})$ is not less than*

$$t_2 - t_1 + \frac{\nu^2}{2} \int_{t_1}^{t_2} \frac{1}{m(t)\sqrt{m(t)^2 - \nu^2}} dt, \quad (4.12)$$

where we set $t_1 := \tilde{x}(0)$ and $t_2 := \tilde{x}(s_0)$.

Proof. We may assume that $t_2 > t_1$, otherwise (4.12) is non-positive. Let $[s_1, s_2]$ be a sub-interval of $[0, s_0]$ such that $\tilde{x}'(s) \neq 0$ on (s_1, s_2) . By (4.11),

$$L(\tilde{\gamma}|_{[s_1, s_2]}) = s_2 - s_1 = \left| \int_{\tilde{x}(s_1)}^{\tilde{x}(s_2)} \frac{m(t)}{\sqrt{m(t)^2 - \nu^2}} dt \right|.$$

Since $\tilde{x}'(s) \neq 0$ for all $s \in (s_1, s_2)$ with $\tilde{x}(s) \in [t_1, t_2]$, we may choose the numbers s_1 and s_2 in such a way that $\tilde{x}(s_1) = t_1$ and $\tilde{x}(s_2) = t_2$ or that $\tilde{x}(s_1) = t_2$ and $\tilde{x}(s_2) = t_1$. Thus, we see that

$$L(\tilde{\gamma}) \geq \int_{t_1}^{t_2} \frac{m(t)}{\sqrt{m(t)^2 - \nu^2}} dt. \quad (4.13)$$

Since

$$\frac{m(t)}{\sqrt{m(t)^2 - \nu^2}} \geq 1 + \frac{\nu^2}{2m(t)\sqrt{m(t)^2 - \nu^2}},$$

we have, by (4.13),

$$L(\tilde{\gamma}) \geq t_2 - t_1 + \frac{\nu^2}{2} \int_{t_1}^{t_2} \frac{1}{m(t)\sqrt{m(t)^2 - \nu^2}} dt.$$

□

The next lemma is well-known in the case of the cut locus of a point (see [B]), Although it can be proved similarly, we here give a proof of the lemma totally different from it.

Lemma 4.4 *For any $q \in \text{Cut}(\partial X) \cap (X \setminus \partial X)$ and any $\varepsilon > 0$, there exists a point in $\text{Cut}(\partial X) \cap B_\varepsilon(q)$ which admits at least two ∂X -segments.*

Proof. Suppose that the cut point q admits a unique ∂X -segment μ_q to q . Then, q is the first focal point of ∂X along μ_q . For each $p \in \partial X$, we denote by v_p the inward pointing unit normal vector to ∂X at $p \in \partial X$. And let \mathcal{U} be a sufficiently small open neighborhood around $d(\partial X, q)\mu'_q(0)$ in the normal bundle $\mathcal{N}_{\partial X}$ of ∂X , so that there exists a number $\lambda(v_p) \in (0, \infty)$ such that $\exp^\perp(\lambda(v_p)v_p)$ is the first focal point of ∂X for each $\lambda(v_p)v_p \in \mathcal{U}$. Set $k := \liminf_{v_p \rightarrow \mu'_q(0)} \nu(v_p)$, where $\nu(v_p) := \dim \ker(d \exp^\perp)_{\lambda(v_p)v_p}$. Since \mathcal{U} is sufficiently small, we may assume that $\nu(v_p) \geq k$ on $\mathcal{U}_\lambda := \{w/\|w\| \mid w \in \mathcal{U}\}$, which is open in the unit sphere normal bundle of ∂X . It is clear that, for each integer $m \geq 0$, the set $\{v_p \in \mathcal{U}_\lambda \mid \text{rank}(d \exp^\perp)_{\lambda(v_p)v_p} \geq m\}$ is open in \mathcal{U}_λ . Hence, by [IT2, Lemma 1], λ is smooth on the open set $\{v_p \in \mathcal{U}_\lambda \mid \nu(v_p) \leq k\} = \{v_p \in \mathcal{U}_\lambda \mid \nu(v_p) = k\} \subset \mathcal{U}_\lambda$. Since $(d \exp^\perp)_{\lambda(v_p)v_p} : T_{\lambda(v_p)v_p} \mathcal{N}_{\partial X} \rightarrow T_{\exp^\perp(\lambda(v_p)v_p)} X$ is a linear map depending smoothly on $v_p \in \mathcal{U}_\lambda$, there exists a **non-zero** vector field W on \mathcal{U}_λ such that $W_{v_p} \in \ker(d \exp^\perp)_{\lambda(v_p)v_p}$ on \mathcal{U}_λ . Here, we assume that $\ker(d \exp^\perp)_{\lambda(v_p)v_p} \subset T_{v_p} \mathcal{U}_\lambda$ by the natural identification.

Assume that there exists a sequence $\{\mu_i : [0, \ell_i] \rightarrow X\}$ of ∂X -segments convergent to μ_q such that $\mu_i(\ell_i) \in \text{Cut}(\partial X)$ and $\mu_i(\ell_i) \notin \text{Foc}(\partial X)$ along μ_i . Then it is clear that each $\mu_i(\ell_i)$ admits at least two ∂X -segments. Hence, we have proved our lemma in this case.

Assume that $\exp^\perp(\lambda(v_p)v_p) \in \text{Cut}(\partial X)$ for all $v_p \in \mathcal{U}_\lambda$. Let $\sigma(s)$, $s \in (-\delta, \delta)$, be the local integral curve of W on \mathcal{U}_λ with $\mu'_q(0) = \sigma(0)$. Hence, $(d \exp^\perp)_{\lambda(\sigma(s))\sigma(s)}(\sigma'(s)) = 0$ on $(-\delta, \delta)$. By [IT1, Lemma 1], $\exp^\perp(\lambda(\sigma(s))\sigma(s)) = \exp^\perp(\lambda(\sigma(0))\sigma(0)) = q$ holds. Hence q is a point in $\text{Cut}(\partial X)$ admitting at least two ∂X -segments. □

Remark 4.5 Lemma 4.4 holds without curvature assumption on $(X, \partial X)$.

Proposition 4.6 Let $\mu_0 : [0, \infty) \rightarrow X$ be a ∂X -ray guaranteed by the assumption above. If $(\tilde{X}, \partial\tilde{X})$ satisfies

$$\int_0^\infty \frac{1}{m(t)^2} dt = \infty, \quad (4.14)$$

or

$$\liminf_{t \rightarrow \infty} m(t) = 0, \quad (4.15)$$

then, any point of X lies in a unique ∂X -ray. In particular, ∂X is totally geodesic in the case where (4.14) is satisfied.

Proof. Choose any point $q \in X \setminus \partial X$ not lying on μ_0 . Let $\mu_1 : [0, d(\partial X, q)] \rightarrow X$ denote a ∂X -segment with $\mu_1(d(\partial X, q)) = q$. For each $t > 0$, let $\gamma_t : [0, d(q, \mu_0(t))]$ denote a minimal geodesic segment emanating from q to $\mu_0(t)$. From Theorem 2.12 and the triangle inequality, it follows that there exists a generalized open triangle

$$\text{GOT}(\partial\tilde{X}, \hat{\mu}_0(t), \hat{q}) = (\partial\tilde{X}, \hat{\mu}_0(t), \hat{q}; \hat{\gamma}_t, \hat{\mu}_0^{(t)}, \hat{\mu}_1)$$

in \tilde{X} corresponding to the triangle $\text{OT}(\partial X, \mu_0(t), q) = (\partial X, \mu_0(t), q; \gamma_t, \mu_0|_{[0, t]}, \mu_1)$ in X such that

$$d(\partial\tilde{X}, \hat{\mu}_0(t)) = t, \quad d(\partial\tilde{X}, \hat{q}) = d(\partial X, q), \quad (4.16)$$

and

$$L(\hat{\gamma}_t) \leq d(\mu_0(t), q) \leq t + d(q, \mu_0(0)) \quad (4.17)$$

and that

$$\angle(\partial X, q, \mu_0(t)) \geq \angle(\partial\tilde{X}, \hat{q}, \hat{\mu}_0(t)). \quad (4.18)$$

Here $\angle(\partial X, q, \mu_0(t))$ denotes the angle between two sides μ_1 and γ_t joining q to ∂X and $\mu_0(t)$ forming the triangle $\text{OT}(\partial X, \mu_0(t), q)$. From Lemma 4.3, (4.16), and (4.17), we get

$$\begin{aligned} t + d(q, \mu_0(0)) &\geq L(\hat{\gamma}_t) \\ &\geq t - d(\partial X, q) + \frac{\nu_t^2}{2} \int_{d(\partial X, q)}^t \frac{1}{m(t) \sqrt{m(t)^2 - \nu_t^2}} dt. \end{aligned} \quad (4.19)$$

where ν_t denotes the Clairaut constant of $\hat{\gamma}_t$. By (4.19),

$$d(\partial X, q) + d(q, \mu_0(0)) \geq \frac{\nu_t^2}{2} \int_{d(\partial X, q)}^t \frac{1}{m(t)^2} dt. \quad (4.20)$$

First, assume that $(\tilde{X}, \partial\tilde{X})$ satisfies (4.14). Then, it is clear from (4.20) that $\lim_{t \rightarrow \infty} \nu_t = 0$. Hence, by (4.9), we have

$$\lim_{t \rightarrow \infty} \angle(\partial\tilde{X}, \hat{q}, \hat{\mu}_0(t)) = \pi. \quad (4.21)$$

By (4.18) and (4.21), $\gamma_\infty := \lim_{t \rightarrow \infty} \gamma_t$ is a ray emanating from q such that

$$\angle(\gamma'_\infty(0), -\mu'_1(d(\partial X, q))) = \pi.$$

This implies that q lies on a unique ∂X -segment. Therefore, by Lemma 4.4, q lies on a ∂X -ray. Now, it is clear from Lemma 4.2 that ∂X is totally geodesic.

Second, assume that $(\tilde{X}, \partial\tilde{X})$ satisfies (4.15). Then, there exists a divergent sequence $\{t_i\}_{i \in \mathbb{N}}$ such that

$$\lim_{t \rightarrow \infty} m(t_i) = 0. \quad (4.22)$$

From (4.9), we see

$$\nu_i \leq m(t_i), \quad (4.23)$$

where ν_i denotes the Clairaut constant of $\hat{\gamma}_{t_i}$. Hence, by (4.22) and (4.23), $\liminf_{t \rightarrow \infty} \nu_t = 0$ holds. Now, it is clear that there exist a limit geodesic γ_∞ of $\{\gamma_{t_i}\}$ such that γ_∞ is a ray emanating from q and satisfies $\angle(\gamma'_\infty(0), -\mu'_1(d(\partial X, q))) = \pi$. Therefore, by Lemma 4.4, q lies on a ∂X -ray. \square

By Proposition 4.6, there does not exist a cut point of ∂X . Therefore, it is clear that

Corollary 4.7 *If $(\tilde{X}, \partial\tilde{X})$ satisfies (4.14), or (4.15), then X is diffeomorphic to $[0, \infty) \times \partial X$.*

Furthermore, we may reach stronger conclusion than Corollary 4.7:

Theorem 4.8 *If $(\tilde{X}, \partial\tilde{X})$ satisfies*

$$\int_0^\infty \frac{1}{m(t)^2} dt = \infty,$$

then, for every ∂X -ray $\mu : [0, \infty) \rightarrow X$, the radial curvature K_X satisfies

$$K_X(\sigma_t) = G(\tilde{\mu}(t)) \quad (4.24)$$

for all $t \in [0, \infty)$ and all 2-dimensional linear space σ_t spanned by $\mu'(t)$ and a tangent vector to X at $\mu(t)$. In particular, X is isometric to the warped product manifold $[0, \infty) \times_m \partial X$ of $[0, \infty)$ and $(\partial X, g_{\partial X})$ with the warping function m . Here $g_{\partial X}$ denotes the induced Riemannian metric from X .

Proof. Take any point $p \in \partial X$, and fix it. By Proposition 4.6, we may take a ∂X -ray $\mu : [0, \infty) \rightarrow X$ emanating from $p = \mu(0)$. Suppose that

$$K_X(\sigma_{t_0}) > G(\tilde{\mu}(t_0)) \quad (4.25)$$

for some linear plane σ_{t_0} spanned by $\mu'(t_0)$ and a unit tangent vector v_0 orthogonal to $\mu'(t_0)$. If we denote by $E(t)$ the parallel vector field along μ satisfying $E(t_0) = v_0$, then $E(t)$ is unit and orthogonal to $\mu'(t_0)$ for each t . We define a non-zero vector field $Y(t)$ along μ by $Y(t) := f(t)E(t)$, where f is the solution of the following differential equation

$$f''(t) + K_X(\mu'(t), E(t))f(t) = 0 \quad (4.26)$$

with initial condition $f(0) = 1$ and $f'(0) = 0$. Here $K_X(\mu'(t), E(t))$ denotes the sectional curvature of the plane spanned by $\mu'(t)$ and $E(t)$. It follows from (4.25) and (L-1) in Lemma 4.1 that there exists $t_1 > 0$ such that $f(t_1) = 0$. From (4.26), we get

$$I_{t_1}(Y, Y) = \int_0^{t_1} \frac{d}{dt}(ff')dt = 0. \quad (4.27)$$

Since ∂X is totally geodesic by Proposition 4.6, $A_{\mu'(0)}(E(0)) = 0$. Thus, by (4.27), $\mathcal{I}_{\partial X}^{t_1}(Y, Y) = 0$ holds. On the other hand, $\mathcal{I}_{\partial X}^{t_1}(Y, Y) > 0$ holds, since there is no focal point of ∂X along μ . This is a contradiction. Therefore, we get the first assertion (4.24).

Now it is clear that the map $\varphi : [0, \infty) \times_m \partial X \rightarrow X$ defined by $\varphi(t, q) := \exp^\perp(tv_q)$ gives an isometry from $[0, \infty) \times_m \partial X$ onto X . Here v_q denotes the inward pointing unit normal vector to ∂X at $q \in \partial X$. \square

5 Proof of Theorem 1.2

Throughout this section, let $(X, \partial X)$ be a complete connected Riemannian manifold X with **disconnected** smooth compact **convex** boundary ∂X whose radial curvature is bounded from below by 0. Under the hypothesis, we may assume

$$\partial X = \bigcup_{i=1}^k \partial X_i, \quad k \geq 2.$$

Here each ∂X_i denotes a connected component of ∂X and is compact. Set

$$\ell := \min\{d(\partial X_i, \partial X_j) \mid 1 \leq i, j \leq k, i \neq j\}.$$

Then let $\partial X_1, \partial X_2$ denote the connected components of ∂X satisfying

$$d(\partial X_1, \partial X_2) = \ell.$$

The proof of the next lemma is standard:

Lemma 5.1 *Let μ denote a minimal geodesic segment in X emanating from ∂X_1 to ∂X_2 . Then, there does not exist any other ∂X -segment to $\mu(\ell/2)$ than $\mu|_{[0, \ell/2]}$ and $\mu|_{[\ell/2, \ell]}$. Furthermore, each midpoint $\mu(\ell/2)$ is not a focal point of ∂X along μ .*

Hereafter, the half plane

$$\mathbb{R}_+^2 := \{\tilde{p} \in \mathbb{R}^2 \mid \tilde{x}(\tilde{p}) \geq 0\}$$

with Euclidean metric $d\tilde{x}^2 + d\tilde{y}^2$ will be used as the model surface for $(X, \partial X)$.

Lemma 5.2 *Any point in X lies on a minimal geodesic segment emanating from ∂X_1 to ∂X_2 of length ℓ . In particular, ∂X consists of ∂X_1 and ∂X_2 .*

Proof. Since X is connected, it is sufficient to prove that the subset \mathcal{O} of X is open and closed, where \mathcal{O} denotes the set of all points $r \in X$ which lies on a minimal geodesic segment emanating from ∂X_1 to ∂X_2 of length ℓ . Since it is trivial that \mathcal{O} is closed, we will prove that \mathcal{O} is open.

Choose any point $r \in \mathcal{O}$, and fix it. Thus, r lies on a minimal geodesic segment $\mu_1 : [0, \ell] \rightarrow X$ emanating from ∂X_1 to ∂X_2 . Set $p := \mu_1(\ell/2)$. Let S be the equidistant set from ∂X_1 and ∂X_2 , i.e.,

$$S := \{q \in X \mid d(\partial X_1, q) = d(\partial X_2, q)\}. \quad (5.1)$$

It follows from Lemma 5.1 that $S \cap B_{\varepsilon_1}(p) \subset \text{Cut}(\partial X)$, if $\varepsilon_1 > 0$ is chosen sufficiently small. Choose any point $q \in S \cap B_{\varepsilon_1}(p) \setminus \{p\}$, and also fix it. Let η_i , $i = 1, 2$, denote a ∂X -segment to q such that $\eta_1(0) \in \partial X_1$ and $\eta_2(0) \in \partial X_2$, respectively. Moreover, let $\gamma : [0, d(p, q)] \rightarrow X$ denote a minimal geodesic segment emanating from p to q . Since

$$\angle(\gamma'(0), -\mu'_1(\ell/2)) + \angle(\gamma'(0), \mu'_1(\ell/2)) = \pi,$$

we may assume, without loss of generality, that

$$\angle(\gamma'(0), -\mu'_1(\ell/2)) \leq \pi/2. \quad (5.2)$$

It follows from Theorem 2.5 that there exists an open triangle

$$\text{OT}(\partial \mathbb{R}_+^2, \tilde{p}, \tilde{q}) = (\partial \mathbb{R}_+^2, \tilde{p}, \tilde{q}; \tilde{\gamma}, \tilde{\mu}_1, \tilde{\eta}_1)$$

in \mathbb{R}_+^2 corresponding to the triangle $\text{OT}(\partial X_1, p, q) = (\partial X_1, p, q; \gamma, \mu_1|_{[0, \ell/2]}, \eta_1)$ such that

$$d(\partial \mathbb{R}_+^2, \tilde{p}) = \ell/2, \quad d(\tilde{p}, \tilde{q}) = d(p, q), \quad d(\partial \mathbb{R}_+^2, \tilde{q}) = d(\partial X_1, q), \quad (5.3)$$

and

$$\angle(\gamma'(0), -\mu'_1(\ell/2)) = \angle p \geq \angle \tilde{p}, \quad \angle q \geq \angle \tilde{q}. \quad (5.4)$$

By (5.2) and $\angle p \geq \angle \tilde{p}$ of (5.4), we have

$$\angle \tilde{p} \leq \pi/2. \quad (5.5)$$

Since our model is \mathbb{R}_+^2 , it follows from the two equations $d(\partial \tilde{X}, \tilde{p}) = \ell/2$, $d(\partial \mathbb{R}_+^2, \tilde{q}) = d(\partial X_1, q)$ of (5.3), and (5.5) that

$$d(\partial X_1, q) = d(\partial \mathbb{R}_+^2, \tilde{q}) \leq \ell/2. \quad (5.6)$$

On the other hand, the broken geodesic segment defined by combining η_1 and η_2 is a curve joining ∂X_1 to ∂X_2 . This implies that length of the broken geodesic segment is not less than that of μ_1 . Thus,

$$2L(\eta_1) = L(\eta_1) + L(\eta_2) \geq \ell, \quad (5.7)$$

where $L(\cdot)$ denotes the length of a curve. Since $L(\eta_1) = d(\partial X_1, q)$, we have, by (5.7), that

$$d(\partial X_1, q) \geq \ell/2. \quad (5.8)$$

By (5.6) and (5.8), $d(\partial X_1, q) = d(\partial X_2, q) = \ell/2$. Therefore, we have proved that any point $q \in S \cap B_{\varepsilon_1}(p)$ is the midpoint of a minimal geodesic segment emanating from ∂X_1 to ∂X_2 of length ℓ . Furthermore, by Lemma 5.1, each point of $S \cap B_{\varepsilon_1}(p)$ is not a focal point of ∂X . It is therefore clear that any point sufficiently close to the point $r \in \mathcal{O}$ is a point of \mathcal{O} , i.e., \mathcal{O} is open. \square

Remark 5.3 From Lemmas 5.1 and 5.2, it is clear that

$$\text{Cut}(\partial X) = \{p \in X \mid d(\partial X, p) = \ell/2\} = S \quad (5.9)$$

and that

$$d(\partial X, p) \leq \ell/2 \quad (5.10)$$

for all $p \in X$. Here S is the equidistant set defined by (5.1). Thus, from the proof of Lemma 5.2, we see that $\angle p = \angle q = \pi/2$ holds for all $p, q \in \text{Cut}(\partial X)$.

Lemma 5.4 $\text{Cut}(\partial X)$ is totally geodesic.

Proof. Let p, q be any mutually distinct points of $\text{Cut}(\partial X)$, and fix them. Moreover, let $\gamma : [0, d(p, q)] \rightarrow X$ denote a minimal geodesic segment emanating from p and q . If we prove that $\gamma(t) \in \text{Cut}(\partial X)$ for all $t \in [0, d(p, q)]$, then our proof is complete.

Suppose that

$$\gamma(t_0) \notin \text{Cut}(\partial X) \quad (5.11)$$

for some $t_0 \in (0, d(p, q))$. By (5.9), we have that

$$d(\partial X, \gamma(t_0)) \neq \ell/2, \quad (5.12)$$

and that

$$d(\partial X, p) = d(\partial X, q) = \ell/2. \quad (5.13)$$

The equations (5.10) and (5.12) imply that

$$d(\partial X, \gamma(t_0)) < \ell/2. \quad (5.14)$$

Without loss of generality, we may assume that

$$d(\partial X, \gamma(t_0)) = \min\{d(\partial X, \gamma(t)) \mid 0 \leq t \leq d(p, q)\}. \quad (5.15)$$

By Remark 5.3, (5.11), and (5.15), we obtain the open triangle $\text{OT}(\partial X, p, \gamma(t_0))$ satisfying

$$\angle p = \pi/2, \quad \angle \gamma(t_0) = \pi/2. \quad (5.16)$$

From Theorem 2.5, (5.13), (5.14), and (5.16), we thus get an open triangle $\text{OT}(\partial \mathbb{R}_+^2, \tilde{p}, \tilde{\gamma}(t_0))$ in \mathbb{R}_+^2 corresponding to the triangle $\text{OT}(\partial X, p, \gamma(t_0))$ such that

$$d(\partial \mathbb{R}_+^2, \tilde{p}) = \ell/2, \quad d(\partial \mathbb{R}_+^2, \tilde{\gamma}(t_0)) < \ell/2,$$

and that

$$\angle \tilde{p} \leq \pi/2, \quad \angle \tilde{\gamma}(t_0) \leq \pi/2.$$

This is a contradiction, since our model is \mathbb{R}_+^2 . Therefore, $\gamma(t) \in \text{Cut}(\partial X)$ holds for all $t \in [0, d(p, q)]$. \square

Lemma 5.5 For each $t \in (0, \ell/2)$, the level set $H_i(t) := \{p \in X \mid d(\partial X_i, p) = t\}$, $i = 1, 2$, is totally geodesic, and $H_1(t)$ is totally geodesic for all $t \in (0, \ell)$.

Proof. Take any $t \in (0, \ell/2)$, and fix it. Let p, q be any mutually distinct points in $H_1(t)$, and also fix them. Let $\mu_1, \mu_2 : [0, \ell] \rightarrow X$ denote minimal geodesic segment emanating from ∂X_1 to ∂X_2 and passing through $\mu_1(t) = p, \mu_2(t) = q$, respectively. Thus, we have an open triangle $\text{OT}(\partial X_1, p, q) = (\partial X_1, p, q; \gamma_t, \mu_1|_{[0, t]}, \mu_2|_{[0, t]})$, where $\gamma_t : [0, d(p, q)] \rightarrow X$ denotes a minimal geodesic segment emanating from p to q . If we prove

$$\angle p = \angle q = \pi/2, \quad (5.17)$$

then we see, by similar argument in the proof of Lemma 5.4, that $H_1(t)$ is totally geodesic. Thus, we will prove (5.17) in the following.

By Theorem 2.5, there exists an open triangle

$$\text{OT}(\partial \mathbb{R}_+^2, \tilde{p}, \tilde{q}) = (\partial \mathbb{R}_+^2, \tilde{p}, \tilde{q}; \tilde{\gamma}_t, \tilde{\mu}_1|_{[0, t]}, \tilde{\mu}_2|_{[0, t]})$$

in \mathbb{R}_+^2 corresponding to the triangle $\text{OT}(\partial X_1, p, q)$ such that

$$d(\partial \mathbb{R}_+^2, \tilde{p}) = d(\partial \mathbb{R}_+^2, \tilde{q}) = t, \quad d(\tilde{p}, \tilde{q}) = d(p, q) \quad (5.18)$$

and that

$$\angle p \geq \angle \tilde{p}, \quad \angle q \geq \angle \tilde{q}. \quad (5.19)$$

Since our model is \mathbb{R}_+^2 , the equation $d(\partial \mathbb{R}_+^2, \tilde{p}) = d(\partial \mathbb{R}_+^2, \tilde{q})$ of (5.18) implies that

$$\angle \tilde{p} = \angle \tilde{q} = \pi/2. \quad (5.20)$$

Thus, by (5.19) and (5.20), we have

$$\angle p \geq \pi/2, \quad \angle q \geq \pi/2. \quad (5.21)$$

On the other hand, by Lemma 5.4, $\text{Cut}(\partial X)$ is totally geodesic, i.e., all eigenvalues of the shape operator of $\text{Cut}(\partial X)$ are 0 in the vector normal to $\text{Cut}(\partial X)$. Since the radial vector of any $\text{Cut}(\partial X)$ -segment is parallel to that of a ∂X -segment, $\text{Cut}(\partial X)$ has also non-negative radial curvature. Therefore, we can apply Theorem 2.5 to the open triangle

$$\text{OT}(\text{Cut}(\partial X), p, q) = (\text{Cut}(\partial X), p, q; \gamma_t, \mu_1|_{[t, \ell/2]}, \mu_2|_{[t, \ell/2]}).$$

Thus, by Theorem 2.5, there exists an open triangle

$$\text{OT}(\partial \mathbb{R}_+^2, \hat{p}, \hat{q}) = (\partial \mathbb{R}_+^2, \hat{p}, \hat{q}; \tilde{\gamma}_t, \tilde{\mu}_1|_{[t, \ell/2]}, \tilde{\mu}_2|_{[t, \ell/2]})$$

in \mathbb{R}_+^2 corresponding to the triangle $\text{OT}(\text{Cut}(\partial X), p, q)$ such that

$$d(\partial \mathbb{R}_+^2, \hat{p}) = d(\partial \mathbb{R}_+^2, \hat{q}) = \ell/2 - t, \quad d(\hat{p}, \hat{q}) = d(p, q) \quad (5.22)$$

and that

$$\pi - \angle p \geq \angle \hat{p}, \quad \pi - \angle q \geq \angle \hat{q}. \quad (5.23)$$

As well as above, the equations (5.22) and (5.23) imply $\pi - \angle p \geq \pi/2$ and $\pi - \angle q \geq \pi/2$, since our model is \mathbb{R}_+^2 . Thus, we have

$$\angle p \leq \pi/2, \quad \angle q \leq \pi/2. \quad (5.24)$$

By (5.21) and (5.24), we therefore get (5.17). By the same argument above, one may prove that $H_2(t)$ is also totally geodesic for all $t \in (0, \ell/2)$. Since $H_1(t) = H_2(\ell - t)$, $H_1(t)$ is totally geodesic for all $t \in (0, \ell)$. \square

Theorem 5.6 *Let $(X, \partial X)$ be a complete connected Riemannian manifold X with disconnected smooth compact convex boundary ∂X whose radial curvature is bounded from below by 0. Then, X is isometric to $[0, \ell] \times \partial X_1$ with Euclidean product metric of $[0, \ell]$ and ∂X_1 , where ∂X_1 denotes a connected component of ∂X . In particular, ∂X_1 is the soul of X .*

Proof. Let $\Phi : [0, \ell] \times \partial X_1 \rightarrow X$ denote the map defined by $\Phi(t, p) := \exp^\perp(t v_p)$, where v_p denotes the inward pointing unit normal vector to ∂X_1 at $p \in \partial X_1$. We will prove that the Φ is an isometry. From Lemma 5.2, it is clear that Φ is a diffeomorphism.

Let $\mu_1 : [0, \ell] \rightarrow X$ denote any minimal geodesic segment emanating from ∂X_1 to ∂X_2 , and fix it. Choose a minimal geodesic segment $\mu_2 : [0, \ell] \rightarrow X$ emanating from ∂X_1 to ∂X_2 sufficiently close μ_1 , so that, for each $t \in (0, \ell)$, $\mu_1(t)$ is joined with $\mu_2(t)$ by a unique minimal geodesic segment γ_t . Since each level hypersurface $H_1(t)$ is totally geodesic by Lemma 5.5, γ_t meets μ_1 and μ_2 perpendicularly at $\mu_1(t)$ and $\mu_2(t)$, respectively. Therefore, by the first variation formula,

$$\frac{d}{dt}d(\mu_1(t), \mu_2(t)) = 0,$$

holds for all $t \in (0, \ell)$. Thus, $d(\mu_1(t), \mu_2(t)) = d(\mu_1(0), \mu_2(0))$ holds for all $t \in [0, \ell]$. This implies that

$$\left\| d\Phi_{(t,p)} \left(\frac{\partial}{\partial x_i} \right) \right\| = \left\| d\Phi_{(0,p)} \left(\frac{\partial}{\partial x_i} \right) \right\| \quad (5.25)$$

for all $t \in [0, \ell]$. Here $(x_1, x_2, \dots, x_{n-1})$ denotes a system of local coordinates around $p := \mu_1(0)$ with respect to ∂X_1 . Since

$$d\Phi_{(0,p)} \left(\frac{\partial}{\partial x_i} \right) = \left(\frac{\partial}{\partial x_i} \right)_{(0,p)},$$

we get, by (5.25),

$$\left\| d\Phi_{(t,p)} \left(\frac{\partial}{\partial x_i} \right) \right\| = \left\| \left(\frac{\partial}{\partial x_i} \right)_{(0,p)} \right\| = \left\| \left(\frac{\partial}{\partial x_i} \right)_p \right\|. \quad (5.26)$$

It is clear that

$$d\Phi_{(t,p)} \left(\frac{\partial}{\partial x_i} \right) \perp d\Phi_{(t,p)} \left(\frac{\partial}{\partial x_0} \right), \quad i = 1, 2, \dots, n-1, \quad (5.27)$$

and

$$\left\| d\Phi_{(t,p)} \left(\frac{\partial}{\partial x_0} \right) \right\| = 1 \quad (5.28)$$

for all $t \in [0, \ell]$. Here x_0 denotes the standard local coordinate system for $[0, \ell]$. By (5.26), (5.27), (5.28), Φ is an isometry. \square

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